Theory Compilation and Theory Approximation

Jorge A. Baier

Departamento de Ciencia de la Computación
Pontificia Universidad Católica de Chile

Santiago, Chile

October 18, 2011

Why Theory Compilation/Approximation?

- Propositional theories can encode a wide variety of knowledge.
- But reasoning tasks are computationally expensive.
- In many applications we would like the answer to come up quickly.
- But we *don't care* if we have to spend a long time pre-processing.

The Main Idea

Let:

- Σ be a propositional theory
- P a property that we want to verify on Σ.

Do as follows.

- Preprocess Σ and produce Σ'.
- Check if the property holds for Σ'.
- Hopefully, the property can be checked more quickly in Σ'.

Propositional Subsets with Good Properties

Example propositional subsets with good properties:

- Horn Theories (SAT: polynomial)
- Prime Implicates (Clause entailment: polynomial)

We should be careful:

- Fact 1: not all theories have a Horn-theory equivalent.
- Fact 2: the number of prime implicates of Σ may be *exponential* in the size of Σ.

A Few Definitions

Definition

A literal is a variable or the negation of a variable (e.g. $p, \neg r$)

Definition

A clause is a disjunction of literals.

Definition

A clause is a Horn clause if it contains at most one positive literal.

A polynomial-time algorithm to decide Horn-SAT (part 1)

- We assume clauses are expressed as sets of literals.
- Let ℓ denote the complement of a literal ℓ.
- Let Σ be a set of clauses.

$$\Sigma|\ell = \{C \setminus \{\bar{\ell}\} \mid C \in \Sigma \text{ and } \ell \notin C\}$$

Theorem

A set of positive Horn clauses is satisfiable.

A polynomial-time algorithm to decide Horn-SAT (part 2)

Input: Horn theory Σ.
Output: Whether or not Σ is SAT.

1. **for each** $\ell \in \Sigma$ **do** $\Sigma := \Sigma|\ell$.
2. **if** $\{\} \in \Sigma$ **return** UNSAT
3. **else return** SAT

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Theory Approximation

The following material is based on the following paper:

Bart Selman, Henry A. Kautz: *Knowledge Compilation and Theory Approximation.* Journal of the ACM 43(2): 193-224 (1996)

More Definitions

Definition (Horn Upper/Lower Bound)

Let Σ be a set of clauses. The sets Σ_{lb} and Σ_{ub} of Horn clauses are respectively a Horn lower-bound and a Horn upper-bound of Σ iff:

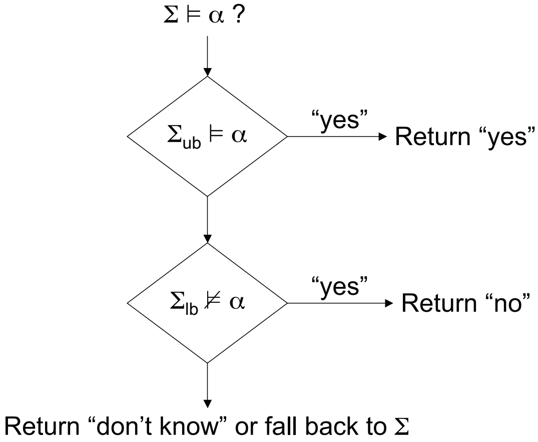
$$\mathcal{M}(\Sigma_{lb}) \subseteq \mathcal{M}(\Sigma) \subseteq \mathcal{M}(\Sigma_{ub}),$$

where $\mathcal{M}(\Sigma)$ denotes the set of models of Σ.

Definition (Horn Greatest Lower Bound)

A Horn lower-bound Σ_{lb} of Σ is a greatest lowerbound if there is no other Horn lower-bound of Σ, Σ' such that $\mathcal{M}(\Sigma_{lb}) \subsetneq \mathcal{M}(\Sigma')$.

Answering Queries Using Bounds



Finding a LUB/GUB is not easy

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Let Σ be a set of clauses. The GLB of Σ is consistent iff Σ is consistent. The LUB of Σ is consistent iff Σ is consistent.

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We conclude that finding a LUB and a GUB is NP-hard.

Horn Strengthenings

Definition

A Horn clause C_H is a Horn-strengthening of a clause C if $C_H \subseteq C$ and there is no Horn clause C' such that $C_H \subsetneq C' \subsetneq C$.

Now we should be able to prove an interesting result:

Lemma

If Σ_H is a Horn theory, and C is a non-tautological clause such that $\Sigma_H \models C$ then Σ entails a Horn strengthening of C.

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Proof.

Follows from the fact that the resolution of two Horn clauses is a Horn clause. □

A GLB is a strengthening

Lemma

Let Σ_{glb} be a GLB of a theory $\Sigma = \{C_1, \ldots, C_n\}$. Then there is a set of clauses $C'_1, \ldots, C'_n$ such that $\Sigma_{glb} \equiv \{C'_1, \ldots, C'_n\}$, where C'_i is a Horn strengthening of C_i.

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By definition of GLB, $\Sigma_{glb} \models \Sigma$, and therefore $\Sigma_{glb} \models C_i$, for $i \in \{1, \ldots, n\}$.

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By the previous lemma, $\Sigma_{glb} \models C'_i$, for some Horn-strengthening C'_i of a $C_i \in \Sigma$.

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Thus $\Sigma_{glb} \models \{C'_1, \ldots, C'_n\} \models \Sigma$.

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By definition of GLB, $\Sigma_{glb} \models \Sigma$, and therefore $\Sigma_{glb} \models C_i$, for $i \in \{1, \ldots, n\}$.
By the previous lemma, $\Sigma_{glb} \models C'_i$, for some Horn-strengthening C'_i of a $C_i \in \Sigma$.
Thus $\Sigma_{glb} \models \{C'_1, \ldots, C'_n\} \models \Sigma$.
Finally, because Σ_{glb} is the *greatest* lower-bound, it must be the case that $\Sigma_{glb} \equiv \{C'_1, \ldots, C'_n\}$.

□

Computing a GLB

GenerateGLB Input: a set of clauses $\Sigma = \{C_1, C_2, \ldots, C_n\}$
Output: a greatest Horn lower-bound of Σ.

1. $L :=$ lexicographically first Horn-strengthening of Σ.
2. **while** true
   - 2.1 $L' :=$ lexicographically next Horn-strengthening of Σ.
   - 2.2 **if** non exists **then exit while**
   - 2.3 **if** $L \models L'$ **then** $L := L'$.
3. remove subsumed clauses from L.
4. **return** L.

Observation: This is an anytime algorithm.

Computing the LUB

Again we first prove a theoretical result

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Proof.

The set of Horn prime implicates is implied by Σ, an thus is a Horn upper-bound. Furthermore, it must be the LUB, because at least one of its clauses subsumes any clause in any Horn upper-bound. □

Generating a LUB

Input: A set of clauses $\Sigma = \Sigma_H \cup \Sigma_N$, where Σ_H is a set of Horn clauses and Σ_N is a set of non-Horn clauses.
Output: A LUB of Σ

a. **while** true

   1. try to choose a clause $C_0 \in \Sigma_H$ and $C_1 \in \Sigma_H \cup \Sigma_N$, such that $C_2 = \text{Resolve}(C_0, C_1)$ is not subsumed by any clause in $\Sigma_H \cup \Sigma_N$.
   2. **if** no such choice is possible **then exit while**
   3. delete from Σ_H and Σ_N any clauses subsumed by C_2.
   4. **if** C_2 is Horn **then** $\Sigma_H := \Sigma_H \cup \{C_2\}$
   5. **else** $\Sigma_N := \Sigma_N \cup \{C_2\}$

b. **return** Σ_H.

Explosion of the LUB

The following theory has exponentially many clauses.

$$\begin{aligned} compSci \land Phil \land Psych &\supset CogSci \\ ReadsMcCarthy &\supset (CompSci \lor CogSci) \\ ReadsDennett &\supset (Phil \lor CogSci) \\ ReadsKosslyn &\supset (Psych \lor CogSci) \end{aligned}$$

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In fact, the LUB is equivalent to the set of 2^3 Horn clauses:

$$(p \wedge q \wedge r) \supset CogSci,$$

with $p \in \{CompSci, ReadsMcCarthy\}$,
$q \in \{Phil, ReadsDennett\}$, $r \in \{Psych, ReadsKosslyn\}$.

Blow up of the LUB

Theorem

There exist clausal theories of size n such that the smallest clausal representation is of size $\mathcal{O}(2^n)$.